

QUICK COMPUTATION OF THE PARITY OF $\Pi(x)$

By H. LIFCHITZ

March 1998 (corrected on October 2001)

Let $\Pi(x)$ = Number of prime numbers below or equal to x .

$|x|$ =integer part of x , and $\Omega(x)$ the number of prime divisors of x (with repetition).

$\Omega(x)$ is additive, $\Omega(1) = \Omega(0) = 0$, $\Omega(p) = 1$ and $\Omega(x.y) = \Omega(x) + \Omega(y)$

$$\boxed{\Omega(x!) = \sum_{P_i \leq x} \left\lfloor \frac{x}{P_i} \right\rfloor + \sum_{P_i \leq x^{1/2}} \left\lfloor \frac{x}{P_i^2} \right\rfloor + \sum_{P_i \leq x^{1/3}} \left\lfloor \frac{x}{P_i^3} \right\rfloor + \dots} \quad (1)$$

Indeed $x! = 2^{\lfloor \frac{x}{2} \rfloor + \lfloor \frac{x}{2^2} \rfloor + \dots} .3^{\lfloor \frac{x}{3} \rfloor + \lfloor \frac{x}{3^2} \rfloor + \dots} .5^{\lfloor \frac{x}{5} \rfloor + \lfloor \frac{x}{5^2} \rfloor + \dots} \dots$, therefore

$$\begin{aligned} \Omega(x!) &= \left\lfloor \frac{x}{2} \right\rfloor + \left\lfloor \frac{x}{2^2} \right\rfloor + \left\lfloor \frac{x}{2^3} \right\rfloor + \dots \\ &+ \left\lfloor \frac{x}{3} \right\rfloor + \left\lfloor \frac{x}{3^2} \right\rfloor + \left\lfloor \frac{x}{3^3} \right\rfloor + \dots \\ &\vdots \quad \quad \quad \vdots \end{aligned}$$

Summing vertically we obtain (1)

$$\boxed{\sum_{P_i \leq x^{1/a}} \left\lfloor \frac{x}{P_i^a} \right\rfloor = \sum_{i=1}^{i=\lfloor \frac{x}{2^a} \rfloor} \Pi \left[\left(\frac{x}{i} \right)^{1/a} \right]} \quad (2)$$

Indeed $\left\lfloor \frac{x}{P_i^a} \right\rfloor = 1$ when $1 \leq \frac{x}{P_i^a} < 2$ or $\frac{x}{2} < P_i^a \leq x$ or $\left(\frac{x}{2}\right)^{1/a} < P_i \leq x^{1/a}$

Therefore
$$\sum_{P_i \leq x^{1/a}} \left| \frac{x}{P_i^a} \right| = \sum_{P_i \leq \left(\frac{x}{2}\right)^{1/a}} \left| \frac{x}{P_i^a} \right| + \Pi \left[x^{1/a} \right] - \Pi \left[\left(\frac{x}{2}\right)^{1/a} \right]$$

afterwards
$$\sum_{P_i \leq x^{1/a}} \left| \frac{x}{P_i^a} \right| = \sum_{P_i \leq \left(\frac{x}{3}\right)^{1/a}} \left| \frac{x}{P_i^a} \right| + \Pi \left[x^{1/a} \right] - \Pi \left[\left(\frac{x}{2}\right)^{1/a} \right] - 2 \Pi \left[\left(\frac{x}{3}\right)^{1/a} \right]$$

and comes (2) continuing the process.

Let
$$\Psi_{\Omega}(x) = \sum_{i=1}^{i=\lfloor \log_2 x \rfloor} \Pi \left(x^{1/i} \right) \quad (3)$$

By moebius inversion we have
$$\Pi(x) = \sum_{i=1}^{i=\lfloor \log_2 x \rfloor} \mu(i) \Psi_{\Omega} \left(x^{1/i} \right) \quad (4)$$

Then we have
$$\sum_{i \leq x} \Psi_{\Omega} \left(\frac{x}{i} \right) = \Omega(x!) \quad (5)$$

because
$$\sum_{i \leq x} \Psi_{\Omega} \left(\frac{x}{i} \right) = \sum_{P_i \leq x} \left| \frac{x}{P_i} \right| + \sum_{P_i \leq \lfloor x^{1/2} \rfloor} \left| \frac{x}{P_i^2} \right| + \sum_{P_i \leq \lfloor x^{1/3} \rfloor} \left| \frac{x}{P_i^3} \right| + \dots$$

Inverting (5) comes
$$\Psi_{\Omega}(x) = \sum_{i \leq x} \mu(i) \Omega \left[\left(\frac{x}{i}\right)! \right] \quad (6)$$

Expanding the right part of (6) we find :

$$\Psi_{\Omega}(x) = \sum_{P_i \leq x} \left| \frac{x}{P_i} \right| - 2 * \sum_{P_i < P_j \leq x} \left| \frac{x}{P_i \cdot P_j} \right| + 3 * \sum_{P_i < P_j < P_k \leq x} \left| \frac{x}{P_i \cdot P_j \cdot P_k} \right| - 4 * \sum_{P_i < P_j < P_k < P_l \leq x} \left| \frac{x}{P_i \cdot P_j \cdot P_k \cdot P_l} \right| + \dots$$

Let $A_1(x) = \sum_{P_i \leq x} \left| \frac{x}{P_i} \right|$, $A_2(x) = \sum_{P_i < P_j \leq x} \left| \frac{x}{P_i \cdot P_j} \right|$, $A_3(x) = \sum_{P_i < P_j < P_k \leq x} \left| \frac{x}{P_i \cdot P_j \cdot P_k} \right|$, etc...

Then
$$\Psi_{\Omega}(x) = A_1(x) - 2.A_2(x) + 3.A_3(x) - 4.A_4(x) + \dots \quad (7)$$

But
$$\sum_{i \leq x} \mu(i) \left| \frac{x}{i} \right| = 1$$

Also let
$$x - 1 = A_1(x) - A_2(x) + A_3(x) - A_4(x) + \dots \quad (8)$$

(7)-(8) result in
$$\Psi_{\Omega}(x) = x - 1 - A_2(x) + 2.A_3(x) - 3.A_4(x) + 4.A_5(x) - \dots \quad (9)$$

We note that $\Psi_{\Omega}(x) = x - 1$ for $1 < x < 6$ and $\Psi_{\Omega}(0) = 0$

So we have
$$\Psi_{\Omega}(x) \equiv x - 1 - A_2(x) - A_4(x) - A_6(x) - \dots [2] \quad (10)$$

But
$$\boxed{\sum_{i \leq x} Q\left(\frac{x}{i}\right) = 2(A_2(x) + A_4(x) + A_6(x) + \dots) + 2x - 1} \quad (11)$$

with $Q(x)$ number of "squarefree" $\leq x$

Indeed let $\prod_k(x)$ = number of "squarefree" $\leq x$ having k prime factors, we also have :

$$A_1(x) = \sum_{i \leq \left\lfloor \frac{x}{2} \right\rfloor} \prod\left(\frac{x}{i}\right), \quad A_2(x) = \sum_{i \leq \left\lfloor \frac{x}{6} \right\rfloor} \prod_2\left(\frac{x}{i}\right), \quad A_3(x) = \sum_{i \leq \left\lfloor \frac{x}{30} \right\rfloor} \prod_3\left(\frac{x}{i}\right), \text{ etc} \dots$$

and $A_1(x) + A_2(x) + A_3(x) + A_4(x) + \dots = \sum_{i \leq \left\lfloor \frac{x}{2} \right\rfloor} \prod\left(\frac{x}{i}\right) + \sum_{i \leq \left\lfloor \frac{x}{6} \right\rfloor} \prod_2\left(\frac{x}{i}\right) + \sum_{i \leq \left\lfloor \frac{x}{30} \right\rfloor} \prod_3\left(\frac{x}{i}\right) + \dots$

But $Q(x) = 1 + \prod(x) + \prod_2(x) + \prod_3(x) + \prod_4(x) + \dots$

So
$$\boxed{A_1(x) + A_2(x) + A_3(x) + A_4(x) + \dots = \sum_{i \leq x} Q\left(\frac{x}{i}\right) - x} \quad (12)$$

Let $SQ(x) = \sum_{i \leq x} Q\left(\frac{x}{i}\right)$

(8) and (12) result in $2(A_2(x) + A_4(x) + A_6(x) + A_8(x) + \dots) = SQ(x) - 2x + 1$, which proves (11)

and also
$$\boxed{SQ(x) = \sum_{i \leq x} Q\left(\frac{x}{i}\right) \equiv 1 [2] \quad \forall x} \quad (13)$$

(10) et (11) result in
$$\boxed{\Psi_\Omega(x) \equiv \frac{SQ(x)-1}{2} [2]} \quad (14)$$

Let $F1(x) = \sum_{i \leq x} \left\lfloor \frac{x}{i} \right\rfloor$, it is easy to show that $F1(x) = 2 \sum_{i=1}^{i=\lfloor \sqrt{x} \rfloor} \left\lfloor \frac{x}{i} \right\rfloor - |\sqrt{x}|^2$

Let
$$\boxed{F1r(x) = \sum_{i \leq \lfloor \sqrt{x} \rfloor} \left\lfloor \frac{x}{i} \right\rfloor}$$
, so we have $F1(x) = 2.F1r(x) - |\sqrt{x}|^2$ and
$$\boxed{F1(x) \equiv |\sqrt{x}| [2]} \quad (15)$$

But
$$\boxed{F1(x) = \sum_{i \leq \lfloor \sqrt{x} \rfloor} SQ\left(\frac{x}{i^2}\right)} \quad (16) \quad \text{and by moebius inversion} \quad \boxed{SQ(x) = \sum_{i \leq \lfloor \sqrt{x} \rfloor} \mu(i).F1\left(\frac{x}{i^2}\right)} \quad (17)$$

So $SQ(x) = \sum_{i \leq \lfloor \sqrt{x} \rfloor} \mu(i) \left[2.F1r\left(\frac{x}{i^2}\right) - \left\lfloor \frac{x}{i^2} \right\rfloor \right] = 2 \sum_{i \leq \lfloor \sqrt{x} \rfloor} \mu(i).F1r\left(\frac{x}{i^2}\right) - \sum_{i \leq \lfloor \sqrt{x} \rfloor} \mu(i). \left\lfloor \frac{x}{i^2} \right\rfloor$

Let
$$\boxed{SQ_1(x) = \frac{SQ(x)-1}{2} = \sum_{i \leq \lfloor \sqrt{x} \rfloor} \mu(i).F1r\left(\frac{x}{i^2}\right) - \left(\frac{\sum_{i \leq \lfloor \sqrt{x} \rfloor} \mu(i) \left\lfloor \frac{x}{i^2} \right\rfloor + 1}{2} \right)} \quad (18)$$

Let $SR(x) = \sum_{i \leq \sqrt{x}} \mu(i) \left| \frac{\sqrt{x}}{i} \right|^2$

$\left| \frac{\sqrt{x}}{i} \right| = 1$ for $\frac{\sqrt{x}}{2} < i \leq \sqrt{x}$, so $SR(x) = \sum_{i \leq \left\lfloor \frac{\sqrt{x}}{2} \right\rfloor} \mu(i) \left| \frac{\sqrt{x}}{i} \right|^2 + \sum_{i=\left\lfloor \frac{\sqrt{x}}{2} \right\rfloor+1}^{i=\sqrt{x}} \mu(i)$

$\left| \frac{\sqrt{x}}{i} \right| = 2$ for $\frac{\sqrt{x}}{3} < i \leq \frac{\sqrt{x}}{2}$, so $SR(x) = \sum_{i \leq \left\lfloor \frac{\sqrt{x}}{3} \right\rfloor} \mu(i) \left| \frac{\sqrt{x}}{i} \right|^2 + 4 \sum_{i=\left\lfloor \frac{\sqrt{x}}{3} \right\rfloor+1}^{i=\left\lfloor \frac{\sqrt{x}}{2} \right\rfloor} \mu(i) + \sum_{i=\left\lfloor \frac{\sqrt{x}}{2} \right\rfloor+1}^{i=\sqrt{x}} \mu(i)$

Continuing the process comes $SR(x) = \sum_{i=1}^{i=\sqrt{x}} i^2 \sum_{j=\left\lfloor \frac{\sqrt{x}}{i+1} \right\rfloor+1}^{j=\left\lfloor \frac{\sqrt{x}}{i} \right\rfloor} \mu(j)$

Seeing that $x^2 \equiv 0[4]$ for even x and $x^2 \equiv 1[4]$ for odd x

we have $SR(x) \equiv \sum_{i \text{ odd} \leq \sqrt{x}} \sum_{j=\left\lfloor \frac{\sqrt{x}}{i+1} \right\rfloor+1}^{j=\left\lfloor \frac{\sqrt{x}}{i} \right\rfloor} \mu(j) \quad [4]$

But $\sum_{i=a+1}^{i=b} \mu(i) = M(b) - M(a)$ with $M(x)$ Moebius summatory function

So $SR(x) \equiv M(\sqrt{x}) - M\left(\frac{\sqrt{x}}{2}\right) + M\left(\frac{\sqrt{x}}{3}\right) - M\left(\frac{\sqrt{x}}{4}\right) + \dots [4]$

But $M(x) - M\left(\frac{x}{2}\right) + M\left(\frac{x}{3}\right) - M\left(\frac{x}{4}\right) + \dots = -1$ property of $M(x)$ for $x \geq 2$

So $SR(x) = \sum_{i \leq \sqrt{x}} \mu(i) \left| \frac{\sqrt{x}}{i} \right|^2 \equiv -1[4]$ for $x \geq 4$

And (18) results in $SQ_1(x) \equiv \sum_{i \leq \sqrt{x}} \mu(i) Fr\left(\frac{x}{i^2}\right) [2] \quad (19)$

So $\Psi_\Omega(x) \equiv SQ_1(x) [2] \quad (20)$

And finally with (4) $\prod(x) \equiv \sum_{i=1}^{i=\lfloor \log_2 x \rfloor} \mu(i) SQ_1(x^{1/i}) [2] \quad (21)$

This expression allows to compute the parity of $\prod(x)$ with a complexity in $O(x^{1/2} * \log x)$ and with

a storage space in $O(1)$.